UNAMBIGUOUS POLYHEDRAL GRAPHS*

BY

BRANKO GRÜNBAUM

ABSTRACT

The existence of unambiguous d-polyhedral graphs is established for every d.

1. A graph G is called *d-polyhedral* provided G can be realized by the vertices and edges of a d-dimensional convex polytope [3]. In general, a d-polyhedral graph may be *dimensionally ambiguous*, i.e., it may be also d'-polyhedral for $d' \neq d$ (though this can not occur for $d \leqq 3$ [3]). A polyhedral graph is *unambiguous* provided it is not dimensionally ambiguous, and provided for every two convex polytopes realizing the graph a biunique correspondence exists between their vertices in such a way that a set of vertices of one of the polytopes determines a face of the polytope if and only if the corresponding vertices of the other determine one of its faces.

Recently, Klee [5] disproved one of the conjectures of [3] and established the existence, for every d, of d-polyhedral graphs which are not dimensionally ambiguous. Klee's proof is based on a new condition for d-polyhedrality. The aim of the present paper is to give a simpler proof and a slight sharpening of Klee's result, by proving the following

THEOREM. *For every d there exist unambiguous d-polyhedral graphs.*

The author is indebted to Victor Klee for many long and interesting conversations on polyhedral graphs.

2. Before proving the theorem, we collect some well-known definitions and facts, and state a few easily established assertions.

If P is a d-dimensional convex polytope in Euclidean d-space E^d, F a $(d-1)$-face of P, and A a point, we shall say that A is *beyond F* provided A belongs to the open halfspace which has F in its boundary and which does not meet P.

The following statements are easily established:

(i) If P is a d-dimensional convex polytope and if A is a point of E^d not belonging to P, there is a $(d-1)$-face F of P such that A is beyond F (Weyl [6]).

(ii) If A and B are vertices of a d-dimensional convex polytope P, joined by an edge of P, and if P_0 is the convex hull of the vertices of P different from A,

Received January 3, 1964.

* Research supported in part by the National Science Foundation, U. S. A. (NSF–GP–378).

then either A is beyond some $(d-1)$-face of P_0 incident to B, or P_0 is $(d-1)$-dimensional.

(iii) If an open halfspace H contains at least two vertices of a convex polytope then H contains an edge of the polytope.

(iv) If a vertex V of a convex polytope P is beyond exactly one face F of the convex hull of the other vertices of P, then V is joined by an edge of P to each vertex of F.

For the following notions and facts see Gale [1, 2], Klee [4], and the references given in those papers.

A *cyclic polytope* $C(d,n)$ is the convex hull of n distinct points on the "moment curve" in E^d, $n \geqq d+1$, $d \geqq 2$, given parametrically by $(t, t^2, t^3, \cdots, t^d)$. It is well known that $C(d,n)$ is a d-dimensional polytope with n vertices, which is neighborly in the sense that every $s \leqq [d/2]$ of its vertices determine an $(s-1)$-dimensional face of $C(d,n)$. In particular, for $d \geqq 4$, every pair of vertices of $C(d,n)$ determines an edge. All the $(d-1)$-faces of $C(d,n)$ are $(d-1)$-simplices. Their number $m(d,n)$ is given by

$$m(d,n) = \binom{n - \left[\frac{d+1}{2}\right]}{n-d} + \binom{n - \left[\frac{d+2}{2}\right]}{n-d}$$

Let $\mu(d,n)$ denote the maximal possible number of $(d-1)$-dimensional faces for d-dimensional polytopes with n vertices. Obviously $\mu(d,n) \geqq m(d,n)$; it has been conjectured that equality holds in this relation for all d and $n \geqq d+1$. It is known that $\mu(d,n) = m(d,n)$ if either $n \leqq d+3$ or $n \geqq [(d+1)/2]^2 - 1$.

Let $C(d,n)$ be a cyclic polytope with vertices $\{V_i : i = 1, 2, \cdots, n\}$; beyond each of its $m(d,n)$ $(d-1)$-faces F_j we take a point W_j sufficiently near to the centroid of F_j, in such a way that in the convex hull $K(d,n)$ of

$$\{V_i : i = 1, \cdots, n\} \cup \{W_j : j = 1, \cdots, m(d,n)\}$$

each W_j is joined by an edge only to the vertices V_i incident to F_j (then all the edges of $C(d,n)$ are also edges of $K(d,n)$). We call $K(d,n)$ a Kleetope derived from $C(d,n)$. The graph of vertices and edges of $K(d,n)$ shall be denoted by $K^*(d,n)$, its nodes by V_i^*, $1 \leqq i \leqq n$, and W_j^*, $1 \leqq j \leqq m(d,n)$.

3. We shall prove the theorem by establishing the following assertion:

For all n and d, such that $d \geqq 4$ and $n+1 \geqq \max\{2d,[d/2]^2\}$, the d-polyhedral graph $K^*(d,n)$ of the Kleetope $K(d,n)$ is unambiguous.

Proof. (i) $K^*(d,n)$ is not dimensionally ambiguous. Indeed, since each of the nodes W_j^* is d-valent, $K^*(d,n)$ is not d'-polyhedral for $d' > d$. Assuming $K^*(d,n)$ to be realizable by a $(d-1)$-dimensional polytope P, let P_0 be the convex hull of the vertices V_i, $1 \leqq i \leqq n$, of P corresponding to the nodes V_i^* of $K^*(d,n)$. By the above, P_0 has at most $\mu(d-1,n) = m(d-1,n)$ faces of dimension $d-2$. By (i) above, each vertex W_j of P (corresponding to the node W_j of $K^*(d,n)$ is beyond at least one of the $(d-2)$-faces of P_0. Since no two vertices W_j determine an edge of P, (iii) implies that no two of those vertices may be beyond the same $(d-2)$-face of P_0. Therefore $m(d-1,n) \geqq m(d,n)$, in contradiction to the value of $m(d,n)$ and the assumption $n \geqq 2d-1$. Since $n \geqq 2d-1$ implies also $m(d,n) > m((d-s),n)$ for every $s \geqq 1$, the same reasoning shows that $K^*(d,n)$ is not $(d-s)$-polyhedral. Thus $K^*(d,n)$ is not dimensionally ambiguous.

(ii) Let P' and P'' be two d-dimensional polytopes realizing $K^*(d,n)$, with vertices V_i', W_j' and V_i'', W_j''. Let P_0' be the convex hull of the vertices V_i' of P', and P_0'' the convex hull of the vertices V_i'' of P''. In each of P', P'', the vertex corresponding to the node W_j^* of $K^*(d,n)$ is beyond at least one of the $(d-1)$-faces of P_0' resp. P_0'', and vertices corresponding to different nodes W_j^* are not beyond the same $(d-1)$-face. Since P' and P'' have each at most $m(d,n)$ faces of dimension $d-1$, each vertex W_j' or W_j'' is beyond exactly one $(d-1)$-face of P_0' resp. P_0''. By ((iv) above, that face has as vertices exactly those V_i''s resp. V_i'''s which correspond to nodes V_i^* connected to the given W_j^* by edges of $K^*(d,n)$. Therefore each $(d-1)$-face of P_0', and of P_0'', is a $(d-1)$-simplex, and the correspondence of V_i' and W_j' to V_i'' and W_j'' shows that $K^*(d,n)$ is unambiguous.

4. Remarks. (1) The assertion of §3 can be established for some additional values of d and n. Thus, $K^*(5,6)$ is unambiguous. The argument is similar to the above, with the addition that in the present case $P_0 = C(4,6)$ and therefore all its 3-faces are simplices. It follows that each W_j is beyond at least two 3-faces of P_0, which is impossible since $C(4,6)$ has only 9 such faces. Even the 11-node 5-polyhedral graph, obtained from $K^*(5,6)$ by deleting one of the nodes W_j^*, is not dimensionally ambiguous.

(2) It is some of interest to note that although $K^*(5,6)$ is unambiguous, the graph of the polytope polar to $K(5,6)$ (in E^5) is 4-polyhedral.

(3) For a d-dimensional convex polytope C let $K(C)$ denote the Kleetope derive from C, i.e. the polytope obtained from C by adjoining above each of its $(d-1)$-faces a sufficiently flat pyramid. Let $K^*(C)$ denote the graph of $K(C)$.

Conjecture. For every C, the graph $K^*(C)$ is unambiguous.

REFERENCES

1. Gale, D., 1964, Neighborly and cyclic polytopes, *Proc. Symp. Pure Math.*, **7**, 225–232.

2. Gale, D., 1964, On a number of faces of a convex polytope. *Canad. J. Math.*, **16**, 12–17.

3. Grünbaum, B. and Motzkin, T. S., 1963, On polyhedral graphs, *Proc. Symp. Pure Math.*, **7**, 285–290.

4. Klee, V., 1963, On the number of vertices of a convex polytope, Math. Note No. 304. Boeing Sci. Res. Labs., June, 1963 (42 pages); to appear in the *Canadian J. Math.*

5. Klee, V., A property of *d*-polyhedral graphs, Math. Note No. 319. Boeing Sci. Res. Labs., August, 1963 (7 pages); to appear in the *J. Math. and Mech.*

6. Weyl, H., 1934/35, Elementare Theorie der konvexen Polyeder, *Comm. Math. Helv.* **7**, 290–306.

UNIVERSITY OF WASHINGTON, SEATTLE
AND
THE HEBREW UNIVERSITY OF JERUSALEM